On full-separating sets in graphs

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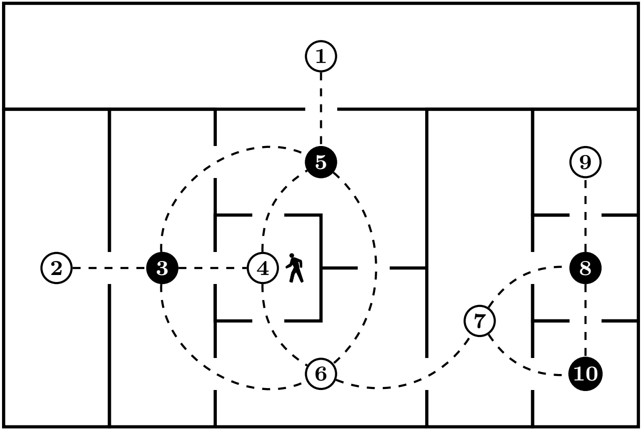
1. Identification problems in graphs : A context for full-separating sets

2. Our results

3. Questions and future research

Identification problems in graphs: A context for full-separating sets...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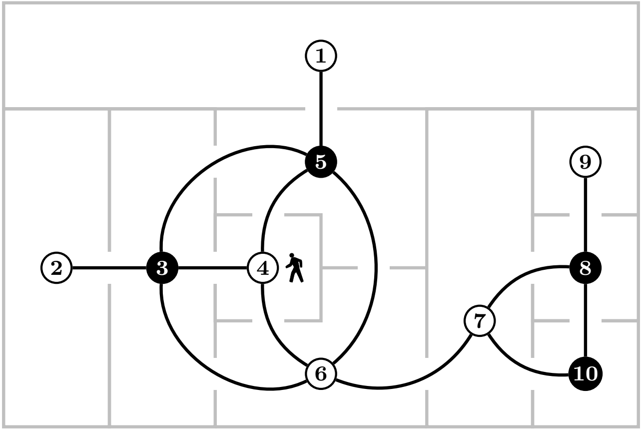
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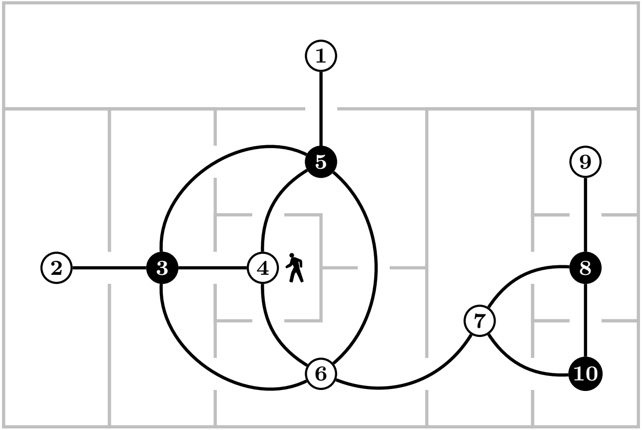
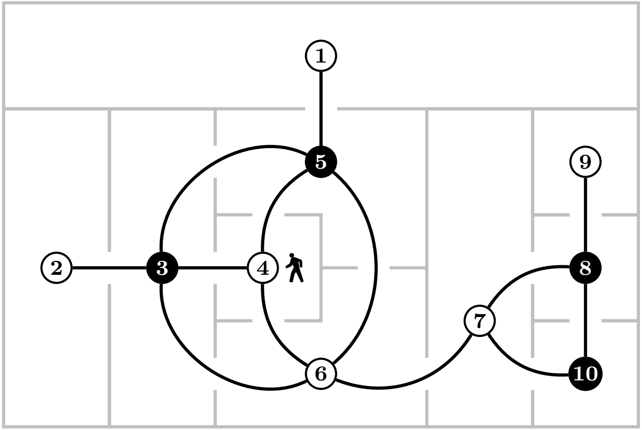


Graph $G = (V, E)$

Open neighborhood:
$N(v) = \{u : uv \in E\}$
$N(7) = \{6, 8, 10\}$

Closed neighborhood:
$N[v] = N(v) \cup \{v\}$
$N[7] = \{6, 8, 10, 7\}$

"**Code**" C = set of black vertices



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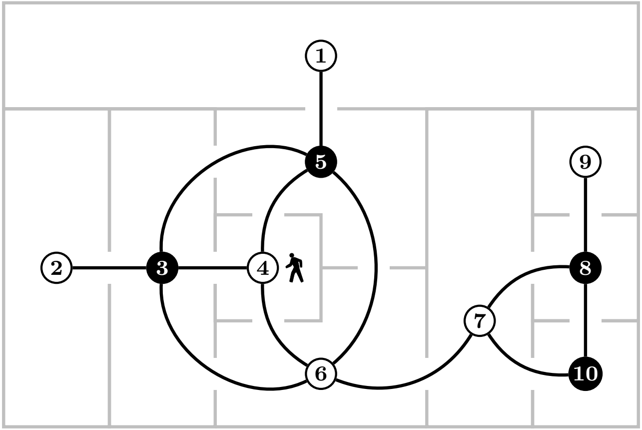
(1) A detector can monitor upto distance 1



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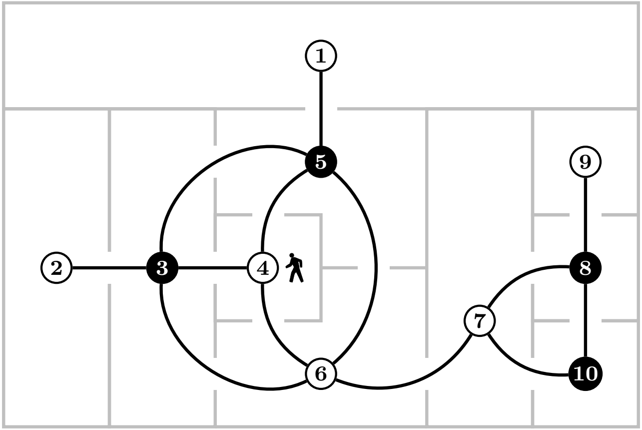
Dominating set: A set $C \subseteq V$ such that $N[v] \cap C \neq \emptyset$ for all $v \in V$



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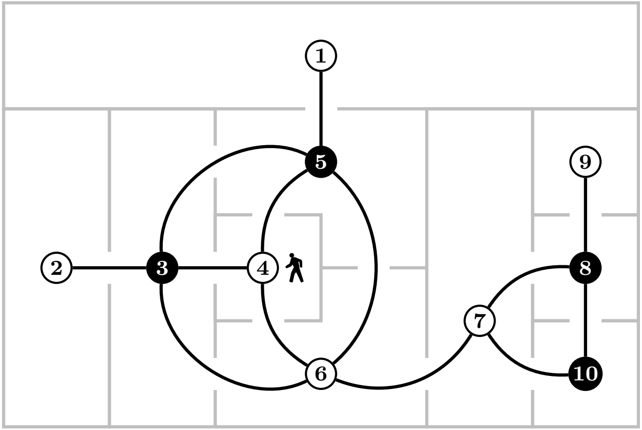

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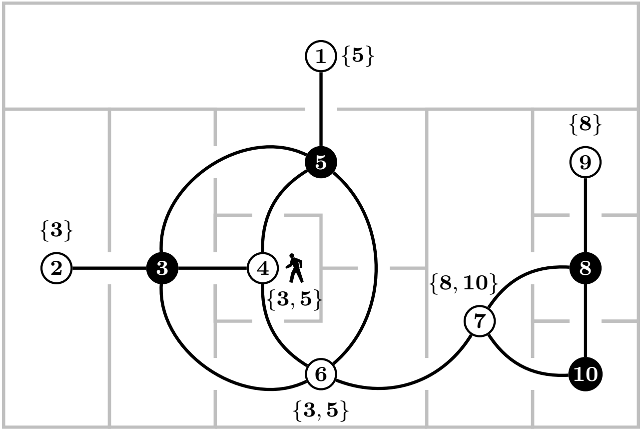
(2) A detector can distinguish between itself and a neighbor



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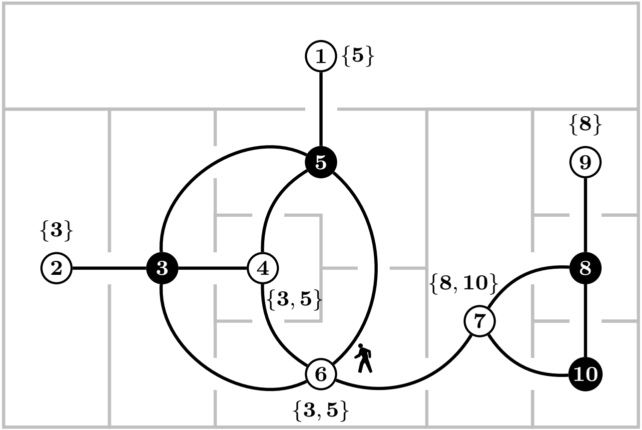

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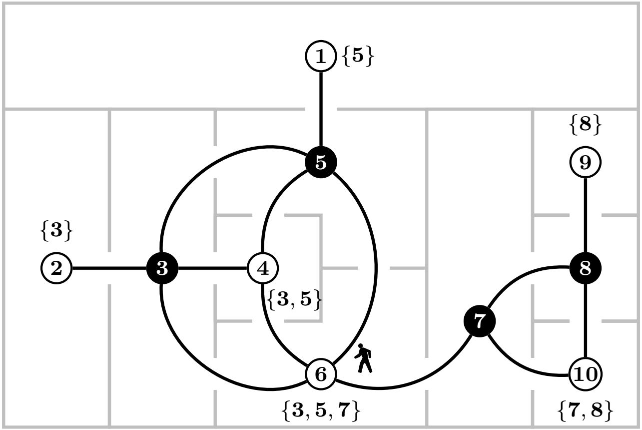

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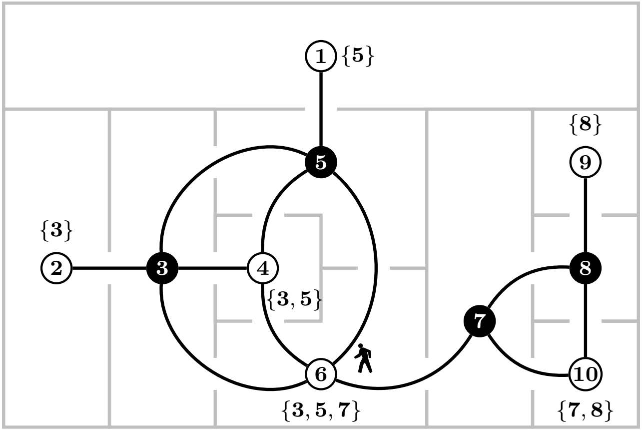

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Separating set (locating): A set $C \subseteq V$ such that
$N(u) \cap C \neq N(v) \cap C \iff \big(N(u) \triangle N(v)\big) \cap C \neq \emptyset$ for all $u, v \in V \setminus C$

A vertex subset C of a graph G is called...

Locating set: if $\big(N(u)\triangle N(v)\big) \cap C \neq \emptyset$ for all $u, v \in V \setminus C$

No faults in detectors: Detectors can distinguish between its vertex and its neighbor

(Slater, 1988)

Closed-separating set: if $\big(N[u]\triangle N[v]\big) \cap C \neq \emptyset$ for all $u, v \in V$

Detector fault type 1: Detector cannot distinguish between itself and its neighbors

(Karpovsky et al., 1998)

Open-separating set: if $\big(N(u)\triangle N(v)\big) \cap C \neq \emptyset$ for all $u, v \in V$

Detector fault type 2: Detector is completely disabled / destroyed

(Honkala et al., 2002 and Seo et al., 2010)

C. and Wagler, 2024:

Full-separating set: if $\big(N[u]\triangle N[v]\big) \cap C = \big(N(u)\triangle N(v)\big) \cap C \neq \emptyset$ for all $u, v \in V$

Detector fault type 1 and detector fault type 2

Full-separating dominating code (FD-code):
A set with full-separating property + dominating property

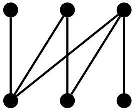
Full-separating total-dominating code (FTD-code):
A set with full-separating property + total-dominating property

Sep	C-Sep		O-Sep		L-Sep		F-Sep	
Code	CD	CTD	OD	OTD	LD	LTD	FD	FTD
adj	$N[u] \triangle N[v]$		$N(u) \triangle N(v)$		$N(u) \triangle N(v)$		$N[u] \triangle N[v]$	
non-adj					$N[u] \triangle N[v]$		$N(u) \triangle N(v)$	
D/TD	$N[u]$	$N(u)$	$N[u]$	$N(u)$	$N[u]$	$N(u)$	$N[u]$	$N(u)$

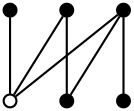
Existence of full-separating sets:
The graph must be twin-free = open-twin-free + closed-twin-free

$\mathrm{X} \in \textsc{Codes} = \{\mathrm{LD}, \mathrm{LTD}, \mathrm{ID}, \mathrm{ITD}, \mathrm{OD}, \mathrm{OTD}, \mathrm{FD}, \mathrm{FTD}\}$

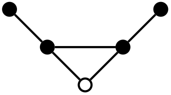
X-number of a graph G: $\gamma^{\mathrm{X}}(G) = \min\{|C| : C \text{ is an X-code of } G\}$



$\gamma^{\mathrm{FTD}}(H_3) = 6$



$\gamma^{\mathrm{FD}}(H_3) = 5$



$\gamma^{\mathrm{FTD}}(B) = \gamma^{\mathrm{FD}}(B) = 4$

Our results...

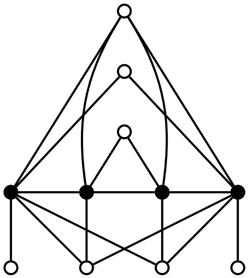
General (tight) bounds...

A twin-free graph G on n vertices with an FD-code C implies

- $n \leq 2^{|C|} - |C|$.
- $\gamma^{\text{FD}}(G) \geq 1 + \lfloor \log_2 n \rfloor$.

A twin-free and isolate-free graph G on n vertices with an FTD-code C implies

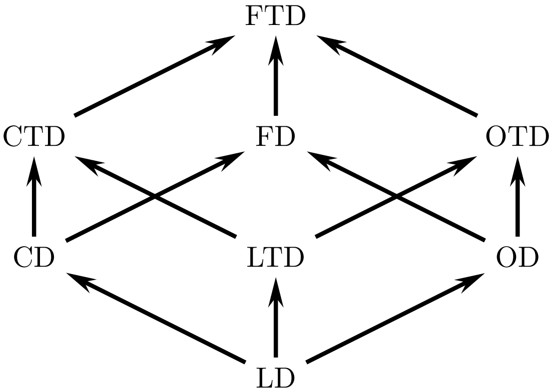
- $n \leq 2^{|C|} - |C| - 1$.
- $\gamma^{\text{FTD}}(G) \geq 1 + \lfloor \log_2(n+1) \rfloor$.



Example of a graph whose FD- and FTD-numbers attain their logarithmic lower bounds.

(NOTE: This is not an example of $n = 2^{|C|} - |C|$, where C is an FD-code)

Relations to other codes...



$X' \longrightarrow X$ stands for $\gamma^{\mathrm{X}'}(G) \leq \gamma^{\mathrm{X}}(G)$

An interesting result...

Theorem (C. and Wagler, 2024)

For a twin-free and isolate-free graph $G = (V, E)$, we have

$$\gamma^{\mathrm{FTD}}(G) - 1 \leq \gamma^{\mathrm{FD}}(G) \leq \gamma^{\mathrm{FTD}}(G).$$

Proof sketch.

Clearly, $\gamma^{\mathrm{FD}}(G) \leq \gamma^{\mathrm{FTD}}(G)$ since any FTD-code is also an FD-code.

To prove the other inequality, take a minimum FD-code C of G.

Notice that there exists at most one $v \in V$ such that $N(v) \cap C = \emptyset$. Thus C is "almost" total-dominating.

To turn C into an FTD-code, we may have to include in C one neighbor of v. This implies $\gamma^{\mathrm{FTD}}(G) \leq |C| + 1 = \gamma^{\mathrm{FD}}(G) + 1$.

Computational complexities...

FD-Code

Input: (G, k): A graph G and a positive integer k.

Problem: Does there exist an FD-code C of G with $|C| \leq k$?

FTD-Code

Input: (G, k): A graph G and a positive integer k.

Problem: Does there exist an FTD-code C of G with $|C| \leq k$?

FD = FTD – 1

Input: A graph G and an integer k.

Problem: Is $\gamma^{\mathrm{FTD}}(G) = k$ and $\gamma^{\mathrm{FD}}(G) = k - 1$?

Computational complexities...

FD-Code

NP-hard!

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FD = FTD − 1

NP-hard!

Input: A graph G and an integer k.

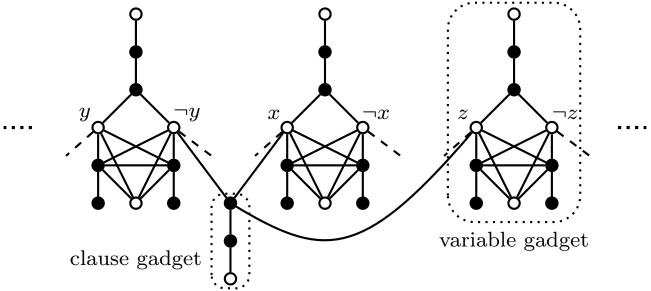
Problem: Is $\gamma^{\text{FTD}}(G) = k$ and $\gamma^{\text{FD}}(G) = k - 1$?

Theorem (C. and Wagler, 2024)

FTD-Code *is NP-complete.*

Proof sketch.

Reduction from 3-SAT with formula ψ on n variables and m clauses.
E.g. $\psi = (x \vee \neg y \vee z) \wedge (\neg x \vee \neg z \vee w) \wedge (\neg y \vee z \vee \neg w)$.



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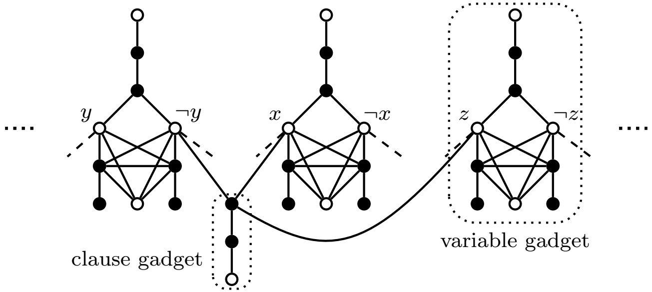
ψ satisfiable $\iff$ $(G^{\psi}, k = 7n + 2m)$ is YES-instance of FTD-Code



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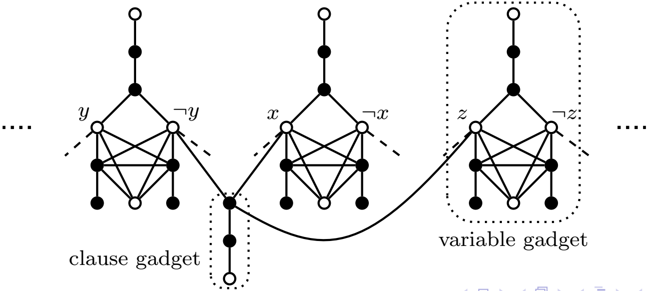
ψ satisfiable $\implies \exists C$ such that $|C| = \gamma^{\mathrm{FTD}}(G^{\psi}) = k$



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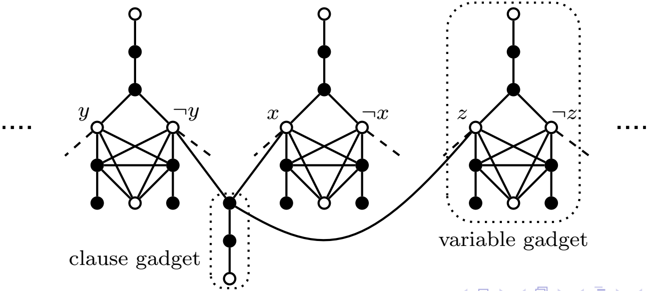

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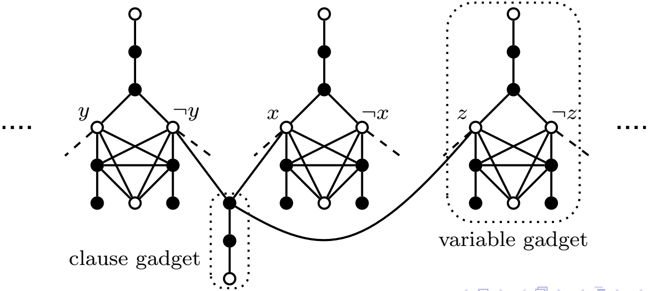
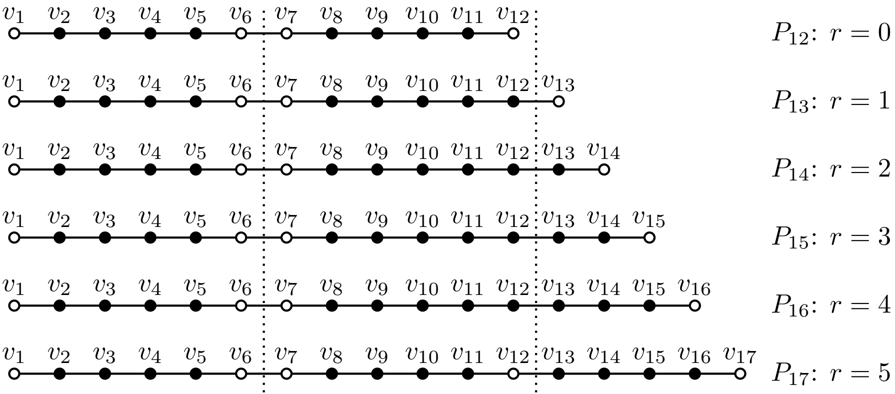
$\boxed{\psi \text{ satisfiable } \iff (G^\psi, 7n+2m-1) \text{ is YES-instance of FD-Code}}$



Theorem (C. and Wagler, 2024)

Let G be either a path P_n for $n \geq 4$ or a cycle C_n for $n \geq 5$. Moreover, let $n = 6q + r$ for non-negative integers q and $r \in [0, 5]$. Then

$$\gamma^{\mathrm{FD}}(G) = \gamma^{\mathrm{FTD}}(G) = \begin{cases} 4q + r, & \text{if } r \in [0, 4]; \\ 4q + 4, & \text{if } r = 5. \end{cases}$$

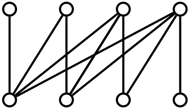


Results on some other selected graph classes...

A subclass of bipartite graphs:

For a half-graph H_k with $k \geq 3$, we have

$$\gamma^{\mathrm{FTD}}(H_k) = 2k \quad \text{and} \quad \gamma^{\mathrm{FD}}(H_k) = 2k - 1.$$

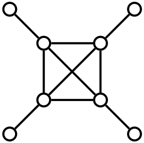


H_4

A subclass of split graphs:

For a thin headless spider S_k with $k \geq 4$, we have

$$\gamma^{\mathrm{FTD}}(S_k) = 2k - 1 \quad \text{and} \quad \gamma^{\mathrm{FD}}(S_k) = 2k - 2.$$

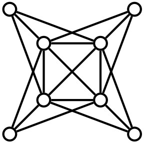


S_4

A subclass of split graphs:

For a thick headless spider $\overline{S}_k$ with $k \geq 4$, we have

$$\gamma^{\mathrm{FTD}}(\overline{S}_k) = 2k - 2 = \gamma^{\mathrm{FD}}(\overline{S}_k).$$



$\overline{S}_4$

Questions and future research...

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- Study bounds and other combinatorial aspects of the FD- and FTD-codes on other well-known graphs classes.
- Study the new code numbers ($\gamma^{\mathrm{FD}}(\cdot)$ and $\gamma^{\mathrm{FTD}}(\cdot)$) with respect to other previously introduced code numbers like $\gamma^{\mathrm{LD}}(\cdot)$, $\gamma^{\mathrm{OTD}}(\cdot)$ etc.
- Explore the algorithmic aspects of FD-Code and FD-Code more.

Thank you!